

ON GROUPS WITH PROPERTY  $(T_{\ell_p})$ 

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ABSTRACT. Let  $1 < p < \infty, p \neq 2$ . Property  $(T_{\ell_p})$  for a second countable locally compact group  $G$  is a weak version of Kazhdan Property  $(T)$ , defined in terms of the orthogonal representations of  $G$  on  $\ell_p$ . Property  $(T_{\ell_p})$  is characterized by an isolation property of the trivial representation from the monomial unitary representations of  $G$  associated to open subgroups. Connected groups with Property  $(T_{\ell_p})$  are the connected groups with a compact abelianization.

In the case of a totally disconnected group, isolation of the trivial representation from the quasi-regular representations associated to open subgroups suffices to characterize Property  $(T_{\ell_p})$ . Groups with Property  $(T_{\ell_p})$  share some important properties with Kazhdan groups (compact generation, compact abelianization, ...). Simple algebraic groups over non-archimedean local fields as well as automorphism groups of  $k$ -regular trees for  $k \geq 3$  have Property  $(T_{\ell_p})$ .

In the case of discrete groups, Property  $(T_{\ell_p})$  implies Lubotzky's Property  $(\tau)$  and is implied by Property  $(F)$  of Glasner and Monod. We show that an irreducible lattice  $\Gamma$  in a product  $G_1 \times G_2$  of locally compact groups have Property  $(T_{\ell_p})$ , whenever  $G_1$  has Property  $(T)$  and  $G_2$  is connected and minimally almost periodic. Such a lattice does not have Property  $(T)$  if  $G_2$  does not have Property  $(T)$ .

## 1. INTRODUCTION

A fundamental rigidity property of groups, with a wide range of applications, is the by now classical Kazhdan's Property  $(T)$  defined in [Kazh]. Bader, Furman, Monod, and Gelander introduced in [BFGM] Property  $(T_B)$  for a general Banach space  $B$ , the case where  $B$  is a Hilbert space corresponding to Kazhdan's Property  $(T)$ . It was shown in [BFGM, Theorem 1] that  $T_{L_p([0,1])}$  for some  $1 < p < \infty$  is equivalent to Kazhdan's Property  $(T)$ , at least for second countable locally compact groups. Moreover, Property  $(T)$  implies  $(T_{L_p(X,\mu)})$  for any  $\sigma$ -finite measure  $\mu$  on a standard Borel space  $X$  and any  $1 \leq p < \infty$ .

In this article, we study Property  $(T_{\ell_p})$  for  $1 < p < \infty$  and  $p \neq 2$ , where  $\ell_p$  is the usual Banach space  $\ell_p(\mathbb{N})$  of  $p$ -summable complex-valued sequences. (We prefer to work with *complex* Banach spaces, as

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this is more suited to our approach which is based on unitary representations on complex Hilbert spaces.)

Let  $G$  be a second countable locally compact group. We recall the definition from [BFGM] of Property  $(T_B)$  for  $G$  in the special case  $B = \ell_p$ .

Denote by  $\mathbf{O}(\ell_p)$  the group of linear bijective isometries of  $\ell_p$ . An orthogonal representation of  $G$  on  $\ell_p$  is a homomorphism of  $\pi : G \rightarrow \mathbf{O}(\ell_p)$ , which is continuous in the sense that the mapping  $G \rightarrow \ell_p, g \mapsto \pi(g)f$  is continuous for every  $f \in \ell_p$ . A sequence  $(f_n)_n$  in  $\ell_p$  is said to be a sequence of almost vectors if  $\|f_n\| = 1$  and

$$\lim_n \|\pi(g)f_n - f_n\| = 0, \quad \text{uniformly on compact subsets of } G.$$

Let  $\pi^* : G \rightarrow \mathbf{O}(\ell_q)$  be the dual representation of  $\pi$  on  $\ell_q$ , where  $q$  is the conjugate exponent of  $p$ . Denote by  $\ell_p^{\pi(G)}$  and  $\ell_q^{\pi^*(G)}$  the closed subspaces of  $G$ -invariant vectors in  $\ell_p$  and  $\ell_q$ , respectively. Let  $\ell'_p(\pi)$  be the annihilator of  $\ell_q^{\pi(G)*}$  in  $\ell_p$ . Then we have a decomposition into  $G$ -invariant subspaces  $\ell_p = \ell_p^{\pi(G)} \oplus \ell'_p(\pi)$ . The group  $G$  has Property  $(T_{\ell_p})$  if there exists no sequence of almost invariant vectors in  $\ell'_p(\pi)$ .

As we will show (see Section 2), Banach's description from [Bana, Chap. XI] of the group  $\mathbf{O}(\ell_p)$  implies that the orthogonal representations of a group  $G$  on  $\ell_p$  have a simple structure for  $p \neq 2$ .

In order to state our first result, we introduce Property  $(T; \mathcal{R})$  with respect to a set  $\mathcal{R}$  of unitary representations of  $G$ ; our definition can easily be reformulated in terms of the Property  $(T; \mathcal{R})$  introduced in [LuZi, Definition 1.1] for sets of irreducible unitary representations of  $G$ .

**Definition 1.** Let  $\mathcal{R}$  be a set of (equivalence classes of) unitary representations of  $G$  on Hilbert spaces. We say that  $G$  has Property  $(T; \mathcal{R})$  if, the trivial representation  $1_G$  of  $G$  is not weakly contained the direct sum  $\oplus_{\pi \in \mathcal{R}} \pi'$ , where  $\pi'$  denotes the restriction of  $\pi$  to the orthogonal complement of the  $\pi(G)$ -invariant vectors in the Hilbert space of  $\pi$  (that is, if the unitary representation  $\oplus_{\pi \in \mathcal{R}} \pi'$  has no sequence of almost invariant vectors).

Observe that Property  $(T)$  for  $G$  corresponds to the case where  $\mathcal{R}$  is the set of all (equivalence classes of) unitary representations of  $G$ .

Recall that a unitary representation  $\sigma$  of  $G$  is *monomial* if  $\sigma$  is unitarily equivalent to an induced representation  $\text{Ind}_H^G \chi$ , where  $H$  is a closed subgroup of  $G$  and  $\chi : H \rightarrow \mathbf{S}^1$  is a unitary character of  $H$ . Examples of monomial representations are the *quasi-regular* representations  $\lambda_{G/H}$  of  $G$  on  $L_2(G/H)$  and correspond to the case where  $\chi = 1_H$ .

**Theorem 2.** *Let  $G$  be a second countable locally compact group. The following properties are equivalent.*

(i)  $G$  has Property  $(T_{\ell_p})$  for some  $1 < p < \infty, p \neq 2$ .

(ii)  $G$  has Property  $(T; \mathcal{R}_{\text{mon}})$ , where  $\mathcal{R}_{\text{mon}}$  is the set of monomial unitary representations  $\text{Ind}_H^G \chi$ , associated to open subgroups  $H$  of  $G$ .

In particular, if  $G$  has Property  $(T_{\ell_p})$  for some  $1 < p < \infty, p \neq 2$ , then  $G$  has Property  $(T_{\ell_p})$  for every  $1 < p < \infty, p \neq 2$ . It is also clear from the previous theorem that Property  $(T)$  implies Property  $(T_{\ell_p})$ ; this is also a special case of Theorem A in [BFGM], as mentioned above. As we will see later, Property  $(T_{\ell_p})$  is strictly weaker than Property  $(T)$ .

When  $G$  is connected, the only open subgroup of  $G$  is  $G$  itself and  $\mathcal{R}_{\text{mon}}$  therefore coincides with the group of unitary characters of  $G$ , that is, with the Pontrjagin dual of the abelianization  $G/[\overline{G}, G]$ . The following corollary is therefore an immediate consequence of Theorem 2.

**Corollary 3.** *A connected locally compact second countable group  $G$  has Property  $(T_{\ell_p})$  for  $1 < p < \infty, p \neq 2$ , if and only if its abelianization  $G/[\overline{G}, G]$  is compact.*

Our next result shows that, when  $G$  is totally disconnected, isolation of  $1_G$  in the set of quasi-regular representations associated to open subgroups suffices to characterize Property  $(T_{\ell_p})$  for  $p \neq 2$ .

**Theorem 4.** *Let  $G$  be a totally disconnected, second countable locally compact group. The following properties are equivalent.*

- (i)  $G$  has Property  $(T_{\ell_p})$  for some  $1 < p < \infty, p \neq 2$ .
- (ii)  $G$  has Property  $(T; \mathcal{R}_{\text{quasi-reg}})$ , where  $\mathcal{R}_{\text{quasi-reg}}$  is the set of quasi-regular representations  $(\lambda_{G/H}, \ell_2(G/H))$  associated to open subgroups  $H$  of  $G$ .

**Remark 5.** (i) Observe that the previous theorem does not hold, in general, if  $G$  is not totally disconnected, as already the example  $G = \mathbf{R}$  shows.

(ii) The result in the previous theorem can be rephrased in terms of the existence of an appropriate Kazhdan pair  $(Q, \varepsilon)$  as in the case of Property  $(T)$ :  $G$  has Property  $(T_{\ell_p})$  for some  $p \neq 2$  if and only if there exists a compact subset  $Q$  of  $G$  and an  $\varepsilon > 0$  such that

$$\sup_{g \in Q} \|\lambda_{G/H}(g)f - f\| \geq \varepsilon$$

for every open subgroup  $H$  and every unit vector  $f$  in the orthogonal complement of the space of  $G$ -invariant vectors in  $\ell_2(G/H)$ .

(iii) We will give below (Example 14 and 17) examples of discrete groups with Property  $(T_{\ell_p})$  and without Property  $(T)$ ; these examples, which seem to be the first of this kind, show that isolation of trivial representation in the family of all quasi-regular representations does not suffice in order to imply Property  $(T)$ .

Groups with Property  $(T_{\ell_p})$  share some important properties with Kazhdan groups.

**Theorem 6.** *Let  $G$  be second countable locally compact group. Assume that  $G$  has Property  $(T_{\ell_p})$  for some  $1 < p < \infty$ . The following statements hold:*

- (i)  *$G$  is compactly generated.*
- (ii) *The abelianized group  $G/[G, G]$  is compact.*
- (iii) *Every subgroup of finite index in  $G$  and every topological group containing  $G$  as a finite index subgroup has Property  $(T_{\ell_p})$ . (In other words, Property  $(T_{\ell_p})$  only depends on the commensurability class of  $G$ .)*
- (iv) *If amenable and totally disconnected, then  $G$  is compact.*

**Remark 7.** (i) It follows from the previous theorem that, for instance, (abelian or non-abelian) free groups as well as the groups  $SL_n(\mathbf{Q})$  do not have Property  $(T_{\ell_p})$ .

(ii) Property  $(T_{\ell_p})$  for  $p \neq 2$  is not inherited by lattices, even in the totally disconnected case. Indeed,  $SL_2(\mathbf{Q}_l)$  has Property  $(T_{\ell_p})$  for  $p \neq 2$  (see Example 9 below), whereas torsion-free discrete subgroups in  $SL_2(\mathbf{Q}_l)$  are free groups (see Chap. II, Théorème 5 in [Ser1]).

The next result will provide us with a class of examples of totally disconnected non discrete groups with Property  $(T_{\ell_p})$  for  $p \neq 2$  and without Property  $(T)$ .

A locally compact group  $G$  has the Howe-Moore property if, for every unitary representation  $\pi$  of  $G$  without non-zero invariant vectors, the matrix coefficients of  $\pi$  are in  $C_0(G)$ . For an extensive study of groups with this property, see [CCLTV].

**Theorem 8.** *Let  $G$  be a totally disconnected group, second countable locally compact group with the Howe-Moore property. Assume that  $G$  is non-amenable. Then  $G$  has Property  $(T_{\ell_p})$  for every  $1 \leq p < \infty, p \neq 2$ .*

**Example 9.** (i) Let  $k$  be a non archimedean local field,  $\mathbb{G}$  a simple linear algebraic group over  $k$  and  $G = \mathbb{G}(k)$  the group of  $k$ -points in  $\mathbb{G}$  (an example is  $G = SL_n(\mathbf{Q}_l)$  for  $n \geq 2$ , where  $\mathbf{Q}_l$  is the field of  $l$ -adic numbers for a prime  $l$ ). Then  $G$  has the Howe-Moore property (see Theorem 5.1 in [HoMo]). Moreover,  $G$  is amenable if and only if  $G$  is compact. So,  $G$  has Property  $(T_{\ell_p})$  for  $p \neq 2$ . Observe that, if  $k - \text{rank}(\mathbb{G}) = 1$ , then  $G$  does not have Property  $(T)$ ; see Remark 1.6.3 in [BeHV]. This is, for instance, the case for  $G = SL_2(\mathbf{Q}_l)$ .

(ii) Let  $G = \text{Aut}(T)$  be the group of color preserving automorphisms of a  $k$ -regular tree  $T$  for  $k \geq 3$  or of a bi-regular tree of type  $(m, n)$  for  $m, n \geq 3$ . Then  $G$  is a totally disconnected locally compact group and, as shown in [LuMo],  $G$  has the Howe-Moore property. Since  $G$  is non-amenable, it has Property  $(T_{\ell_p})$  for  $p \neq 2$ . Observe that  $G$  does not have Property  $(T)$ .

We turn now to discrete groups. The examples of discrete groups with Property  $(T_{\ell_p})$  for  $p \neq 2$  and without Property  $(T)$  we give below

are related either to Lubotzky's Property  $(\tau)$  or to the Glasner-Monod Property  $(F)$ .

Recall that a discrete group  $\Gamma$  has Property  $(\tau)$ , if  $\Gamma$  has Property  $(T; \mathcal{R}_{fi})$ , with respect to the set  $\mathcal{R}_{fi}$  of regular representations  $\lambda_{\Gamma/H}$  associated to subgroups  $H$  of finite index ([Lubo, Definition 4.3.1]). The following result is an immediate consequence of Theorem 4.

**Proposition 10.** *Every discrete group with Property  $(T_{\ell_p})$  has Lubotzky's Property  $(\tau)$ . ■*

**Remark 11.** Property  $(\tau)$  does not imply Property  $(T_{\ell_p})$  for  $p \neq 2$ . Indeed, by Theorem 6, every countable discrete group with Property  $(T_{\ell_p})$  is finitely generated. However, there are groups with Property  $(\tau)$  which are not finitely generated. An example of such a group is  $\Gamma = SL_n(\mathbf{Z}[1/\mathcal{P}])$  for  $n \geq 2$ , where  $\mathcal{P}$  is an infinite set of primes not containing all primes and  $\mathbf{Z}[1/\mathcal{P}]$  denotes the ring of rational numbers whose denominators are only divisible by primes from  $\mathcal{P}$  (see the remarks after Corollary 2.7 in [LuZi]).

We prove Property  $(T_{\ell_p})$  for the following class of lattices. Recall that a lattice  $\Gamma$  in a product  $G_1 \times G_2$  of locally compact groups is irreducible if the natural projections of  $\Gamma$  to  $G_1$  and  $G_2$  are dense. A locally compact group is minimally almost periodic if it has no non-trivial finite dimensional unitary representation.

**Theorem 12.** *Let  $G_1, G_2$  be locally compact second countable groups and  $\Gamma$  an irreducible lattice in  $G = G_1 \times G_2$ . Assume that  $G_1$  has Property  $(T)$  and that  $G_2$  is connected and minimally almost periodic. Then  $\Gamma$  has Property  $(T_{\ell_p})$  for  $p \neq 2$ .*

**Remark 13.** (i) Property  $(\tau)$  was established in [LuZi, Corollary 2.6] (see also Corollary of Theorem A in [BeLo]) for the groups  $\Gamma$  appearing in Theorem 12, without the connectedness assumption on  $G_2$ .

(ii) If  $G_2$  does not have Property  $(T)$  then neither does  $\Gamma$ , since Property  $(T)$  is inherited by lattices ([BeHV, Theorem 1.7.1]).

**Example 14.** Examples of groups  $\Gamma$  as in Theorem 12 are, for instance, lattices in  $SO(n, 2) \times SO(n+1, 1)$  for  $n \geq 3$ . (Observe that Theorem 12 still applies when the connected component of  $G_2$  has finite index, in view of Theorem 6.iii.) Such lattices can be obtained by the following well-known construction. Let  $\mathbf{G} = SO(q)$  be the orthogonal group of the quadratic form  $q(x) = x_1^2 + \cdots + x_n^2 - x_{n+1}^2 - \sqrt{2}x_{n+2}^2$ . Then  $\mathbf{G}(\mathbf{R}) \cong SO(n, 2)$ . Let  $\sigma$  be the non trivial field automorphism of  $\mathbf{Q}(\sqrt{2})$  and  $\mathbf{G}^\sigma = SO(q^\sigma)$  the orthogonal group of the conjugate form  $q^\sigma$ . Then  $\mathbf{G}^\sigma(\mathbf{R}) \cong SO(n+1, 1)$  and  $\Gamma = \mathbf{G}(\mathbf{Z}[\sqrt{2}])$  embeds as an irreducible lattice in  $\mathbf{G}^\sigma(\mathbf{R}) \times \mathbf{G}(\mathbf{R})$  by means of the mapping  $\gamma \mapsto (\gamma^\sigma, \gamma)$ .

Another class of groups with Property  $(T_{\ell_p})$  are the groups with Property  $(F)$  of Glasner and Monod. We first recall how this later property is defined.

A continuous action of a locally compact group  $G$  on a discrete countable space  $X$  is said to be amenable if the natural representation of  $G$  on  $\ell_2(X)$  weakly contains  $1_X$ . One should mention that this notion, extensively studied in [Eyma] in the case of (non necessarily discrete) homogeneous spaces, is different from Zimmer's notion of amenable group action from Section 4.3 in [Zimm]. A discrete group  $\Gamma$  has Property  $(F)$  if every amenable action of  $\Gamma$  on a countable space  $X$  has a fixed point ([GlMo, Definition 1.3]). It turns out that Property  $(F)$  implies Property  $(T_{\ell_p})$  for  $p \neq 2$ .

**Proposition 15.** *Let  $\Gamma$  be a discrete countable group with the Glasner-Monod Property  $(F)$ . Then  $\Gamma$  has Property  $(T_{\ell_p})$  for  $p \neq 2$ .*

**Remark 16.** Property  $(T_{\ell_p})$  does not imply Property  $(F)$ . Indeed, a group with Property  $(F)$  has no non-trivial finite quotient. However there are groups, such as  $SL_3(\mathbf{Z})$ , which have Property  $(T)$  and hence Property  $(T_{\ell_p})$ , and which have non-trivial finite quotients.

**Example 17.** It follows from Proposition 15 that the examples given in [GlMo] of groups with Property  $(F)$  and without Property  $(T)$  are at the same time examples of groups with Property  $(T_{\ell_p})$  for  $p \neq 2$  and without Property  $(T)$ . We briefly recall their construction.

A remarkable feature of Property  $(F)$  is that the class of groups with this property is preserved by finite free products (Lemma 3.1 of [GlMo]). The free product  $\Gamma$  of two non-trivial groups with Property  $(F)$  therefore has Property  $(T_{\ell_p})$  for  $p \neq 2$ . Observe that  $\Gamma$  does not have Property  $(T)$ , since  $\Gamma$  acts on a tree without fixed point. It remains to give examples of groups with Property  $(F)$ . The examples given in [GlMo] are infinite simple Kazhdan groups or non amenable groups for which every proper subgroup is finite ("Tarski monsters"). Examples of the first kind of groups were constructed by Gromov (Corollary 5.5.E in [Grom]) as quotients of hyperbolic groups with Property  $(T)$ ; for another construction, see Corollary 21 in [CaRe]. Examples of the second kind of groups were given by Ol'shanskii [Olsh].

This paper is organized as follows. In Section 2 contains some basic remarks on the structure of orthogonal group representations on  $\ell_p$  for  $p \neq 2$ . Theorems 2 and 4 are proved in Section 3 and Theorems 6 and 8 in Section 4. Section 5 is devoted to discrete groups with Property  $(T_{\ell_p})$  and contains the proof of Theorem 12 and Propositions 10 and 15.

After this work was completed, we learned of the preprint [Corn2] which contains some overlap with our present work. Consider the following weaker version of Property  $(F)$ , introduced (without name) in

[GIMo, Remark 1.4]: a group  $G$  has Property FM, if every amenable action of  $G$  on a countable space has a finite orbit. Using Theorem 4, it is easy to see that a group has Property  $(T_{\ell_p})$  if and only if it has Property FM and Property  $(\tau)$ . Property FM is studied [Corn2], in an independent way and with a different motivation. It is shown there that the lattices  $\Gamma$  appearing in our Theorem 12 have Property FM. As these groups are known to have Property  $(\tau)$ , this gives a different proof of Property  $(T_{\ell_p})$  for these lattices. Moreover, in connection with our Example 9.i, it is proved in [Corn2] that semisimple algebraic groups over non-archimedean local fields have Property FM.

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## 2. ORTHOGONAL REPRESENTATIONS ON $\ell_p$ FOR $p \neq 2$

We begin with some preliminary remarks on permutation representations of topological groups twisted by a cocycle with values in  $\mathbf{S}^1$ . Let  $G$  be a topological group. Let  $X$  be a discrete space equipped with a  $G$ -action. We assume that this action is continuous, or, equivalently, that the stabilizers of points in  $X$  are open subgroups of  $G$ . Let  $c : G \times X \rightarrow \mathbf{S}^1$  be a continuous cocycle with values in  $\mathbf{S}^1$ ; thus,  $c$  satisfies the cocycle relation:

$$c(g_1 g_2, x) = c(g_1, g_2 x) c(g_2, x), \quad \text{for all } g_1, g_2 \in G, x \in X. \quad (*)$$

We associate to the  $G$ -action and the cocycle  $c$  the permutation representation twisted by  $c$ , which is the continuous representation of  $G$  on  $\ell_2(X)$ , denoted by  $\lambda_X^c$  and defined by the formula

$$\lambda_X^c(g)(f)(x) = c(g^{-1}, x) f(g^{-1}(x)), \quad \text{for all } g \in G, f \in \ell_2(X), x \in X.$$

The following lemma is a very special case of Mackey's imprimitivity theorem (see Theorem 3.10 in [Mack]).

**Lemma 18.** *Assume that  $G$  acts transitively on  $X$ . Let  $x_0 \in X$  and denote by  $H$  the stabilizer of  $x_0$  in  $G$ . Let  $\chi : H \rightarrow \mathbf{S}^1$  be defined by  $\chi(h) = c(h, x_0)$  for all  $h \in H$ . Then  $\chi$  is a unitary character of  $H$  and  $\lambda_X^c$  is unitarily equivalent to the monomial representation  $\text{Ind}_H^G \chi$ .*

**Proof** The fact that  $\chi$  is a homomorphism follows immediately from the cocycle relation  $(*)$ .

Fix a set  $T \subset G$  of representatives for the left cosets of  $H$ . The space  $\ell_2(X)$  is the direct sum  $\bigoplus_{x \in X} V_x$ , where  $V_x$  is the one-dimensional space  $\mathbf{C}\delta_x$ . The restriction of  $\lambda_X^c$  to  $H$  leaves  $V_{x_0}$  invariant, with the corresponding  $H$ -representation given by the character  $\chi$ . Moreover, we have  $\lambda_X^c(t)V_{x_0} = V_{tx_0}$  for all  $t \in T$ . This shows that  $\lambda_X^c$  is equivalent to  $\text{Ind}_H^G \chi$ , by the defining property of induced representations (see

Section 3.3 in [Ser2]; the case of finite groups treated there carries over to induced representations from open subgroups of infinite groups). ■

**Remark 19.** Conversely, as is well-known, every monomial representation of  $G$  associated to an open subgroup  $H$  can be realized as a representation of the form  $\lambda_X^c$  for the action of  $G$  on  $X = G/H$  and a continuous cocycle  $c : G \times G/H \rightarrow \mathbf{S}^1$ . We recall briefly the construction of  $c$ . Choose a section  $s : G/H \rightarrow G$  for the canonical projection  $p : G \rightarrow G/H$ , with  $s(H) = e$ . Define a cocycle  $\alpha : G \times X \rightarrow H$  with values in  $H$ , given by

$$\alpha(g, x) = s(gx)^{-1}gs(x), \quad \text{for all } g \in G, x \in X.$$

Then  $c : G \times G/H \rightarrow \mathbf{S}^1$  is defined by  $c(g, x) = \chi(\alpha(g, x))$ .

The following lemma is an immediate consequence of Lemma 18 and the previous remark.

**Corollary 20.** *Let  $G$  be a topological group.*

(i) *Let  $X$  be a discrete space equipped with a continuous  $G$  action and let  $c : G \times X \rightarrow \mathbf{S}^1$  be a continuous cocycle. The associated representation  $\lambda_X^c$  of  $G$  on  $\ell_2(X)$  is equivalent to a direct sum of monomial representations associated to open subgroups of  $G$ .*

(ii) *Let  $\pi = \bigoplus_{i \in I} \text{Ind}_{H_i}^G \chi_i$  be a direct sum of monomial representations associated to open subgroups  $H_i$  of  $G$ . Set  $X = \coprod_{i \in I} G/H_i$ , the disjoint sum of the  $G/H_i$ 's, with the obvious  $G$ -action. Then  $\pi$  is equivalent to the representation  $\lambda_X^c$  of  $G$  on  $\ell_2(X)$  for a cocycle  $c : G \times X \rightarrow \mathbf{S}^1$ . ■*

Next, we study orthogonal representations of topological groups on  $\ell_p$  for  $p \neq 2$ . We first recall Banach's description of  $\mathbf{O}(\ell_p)$  for  $1 \leq p < \infty, p \neq 2$  from Chapitre XI in [Bana].

Let  $X$  be an infinite countable set and  $\text{Sym}(X)$  the group of all permutations of  $X$ . Let  $U \in \mathbf{O}(\ell_p(X))$ . There exists a unique permutation  $\sigma \in \text{Sym}(X)$  and a unique function  $h : X \rightarrow \mathbf{S}^1$  such that

$$U(f)(x) = h(x)f(\sigma(x)), \quad \text{for all } f \in \ell_p(X), x \in X.$$

(One should observe that Banach's theorem is stated in [Bana] for spaces of *real-valued* sequences; however, the arguments remain valid for complex-valued sequences and yield the result stated above.)

Let  $G$  be a topological group and  $\pi : G \rightarrow \mathbf{O}(\ell_p)$  a continuous orthogonal representation of  $G$  on  $\ell_p = \ell_p(X)$  for  $1 \leq p < \infty, p \neq 2$ .

By Banach's result, there exist mappings

$$\varphi : G \rightarrow \text{Sym}(X) \quad \text{and} \quad c : G \times X \rightarrow \mathbf{S}^1$$

such that

$$\pi(g)(f)(x) = c(g^{-1}, x)f(\varphi(g^{-1})(x)), \quad \text{for all } g \in G, f \in \ell_p(X), x \in X.$$

Since  $\pi$  is a group homomorphism, one checks that  $\varphi$  is also a group homomorphism; so,  $\varphi$  defines an action of  $G$  on  $X$ , which we hereafter



denote simply by  $(g, x) \mapsto gx$ . Moreover,  $c : G \times X \rightarrow \mathbf{S}^1$  satisfies the cocycle relation  $(*)$ .

Observe that  $\{\delta_x : x \in X\}$  is a discrete subset of  $\ell_p(X)$ , equipped with the norm topology. Since  $\pi$  is continuous, it follows that the action of  $G$  on the discrete space  $X$  is continuous. Similarly, one checks that  $c : G \times X \rightarrow \mathbf{S}^1$  is continuous.

In summary, to a continuous orthogonal representation  $\pi$  of  $G$  on  $\ell_p(X)$ ,  $1 < p < \infty, p \neq 2$ , is associated an action of  $G$  on  $X$  with open point stabilizers and a continuous cocycle  $c : G \times X \rightarrow \mathbf{S}^1$ . (It is clear that, conversely, such an action of  $G$  on  $X$  and a continuous cocycle  $c : G \times X \rightarrow \mathbf{S}^1$  define a continuous orthogonal representation of  $G$  on  $\ell_p(X)$ .)

Set  $\pi^2 = \lambda_X^c$ , where  $\lambda_X^c$  is the permutation unitary representation of  $G$  on  $\ell_2(X)$  twisted by  $c$ , as defined above.

Let  $\ell_p^{\pi(G)}$  and  $\ell_2^{\pi^2(G)}$  be the closed subspaces of  $G$ -invariant vectors in  $\ell_p(X)$  and  $\ell_2(X)$ . Let  $\ell'_p(\pi)$  and  $\ell'_2(\pi^2)$  be the  $G$ -invariant complements of  $\ell_p^{\pi(G)}$  and  $\ell_2^{\pi^2(G)}$  as described in the introduction. (Observe that  $\ell'_2(\pi^2)$  is the orthogonal complement of  $\ell_2^{\pi^2(G)}$ .)

The Mazur mappings  $M_{2,p} : \ell_2(X) \rightarrow \ell_p(X)$  and  $M_{p,2} : \ell_p(X) \rightarrow \ell_2(X)$  are the non-linear mappings defined by

$$M_{2,p}(f) = (f/|f|)|f|^{2/p} \quad \text{and} \quad M_{p,2}(f) = (f/|f|)|f|^{p/2}.$$

It is easily checked that  $\pi^2(g) = M_{p,2}\pi(g)M_{2,p}$  and  $\pi(g) = M_{2,p}\pi^2(g)M_{p,2}$  for all  $g \in G$ . The mappings  $M_{2,p}$  and  $M_{p,2}$  are uniformly continuous between the unit spheres of  $\ell_2$  and  $\ell_p$  ([BeLi, Theorem 9.1]). As a consequence, one obtains the following crucial fact, established in Section 4.a of [BFGM].

**Lemma 21.** *There exists a sequence of almost invariant vectors for the restriction of  $\pi$  to  $\ell'_p(\pi)$  if and only if there exists a sequence of almost invariant vectors for the restriction of  $\pi^2$  to  $\ell'_2(\pi^2)$ . ■*

### 3. PROOF OF THEOREMS 2 AND 4

#### Proof of Theorem 2

(i)  $\implies$  (ii) :

Assume that  $G$  does not have Property  $(T_{\ell_p})$ . Then there exists an orthogonal representation  $\pi : G \rightarrow \mathbf{O}(\ell_p)$  such that the restriction of  $\pi$  to  $\ell'_p(\pi)$  has a sequence of almost invariant vectors.

By Lemma 21, the unitary representation  $\pi^2$  on  $\ell_2$  associated to  $\pi$  has a sequence of almost invariant vectors in  $\ell'_2(\pi^2)$ . On the other hand, Corollary 20 shows that  $\pi^2$  is unitary equivalent to a direct sum of monomial representations associated to open subgroups. It follows that  $1_G$  does not have Property  $(T; \mathcal{R}_{\text{mon}})$ .

(ii)  $\implies$  (i) :

Assume that  $1_G$  does not have Property  $(T; \mathcal{R}_{\text{mon}})$ . Thus, there exists a family  $(H_i, \chi_i)_{i \in I}$  of open subgroups  $H_i$  with unitary characters  $\chi_i$  with the following property: if  $\pi^2$  denotes the representation  $\bigoplus_{i \in I} \text{Ind}_{H_i}^G \chi_i$  on  $\mathcal{H} = \bigoplus_{i \in I} \ell_2(G/H_i)$ , there exists a sequence  $(f_n)_n$  of almost invariant vectors in the orthogonal complement  $\mathcal{H}'$  of the space  $\mathcal{H}^{\pi(G)}$  of invariant vectors.

For every  $f \in \mathcal{H}$ , the projection of  $f$  on  $\ell_2(G/H_i)$  is non-zero for at most countably many  $i \in I$ . It follows that we can assume that the set  $I$  is infinite countable. (If  $I$  happens to be finite, we replace  $I$  by  $I \times \mathbf{N}$  and set  $H_{(i,n)} = H_i$ .)

Let  $X = \coprod_{i \in I} G/H_i$ . By Corollary 20,  $\pi^2$  is equivalent to the permutation representation  $\lambda_X^c$  of  $G$  on  $\ell_2(X)$  twisted by a cocycle  $c : G \times X \rightarrow \mathbf{S}^1$ .

We can associate to  $\lambda_X^c$  the orthogonal representation  $\pi : G \rightarrow \mathbf{O}(\ell_p(X))$ , defined by the same formula. Lemma 21 shows that  $\pi$  has a sequence of almost invariant vectors contained in the  $G$ -invariant complement  $\ell'_p(\pi)$  of  $\ell_p(X)^{\pi(G)}$ . Therefore,  $G$  does not have Property  $(T_{\ell_p})$ . ■

The proof of Theorem 4 will be an easy consequence of the following lemma.

**Lemma 22.** *Let  $G$  be a totally disconnected group,  $H$  an open subgroup and  $\chi$  a continuous unitary character of  $H$ . There exists an open subgroup  $L$  of  $G$  contained in  $H$  such that the monomial representation  $\text{Ind}_H^G \chi$  is weakly contained in the quasi-regular representation  $\lambda_{G/L}$ .*

**Proof** Since  $G$ , and hence  $H$ , is totally disconnected, every neighbourhood of the group unit in  $H$  contains a compact open subgroup, by Dantzig's theorem (see, e.g., Theorem 7.7 in [HeRo]). By continuity of  $\chi$ , there exists a compact open subgroup  $K$  of  $H$  such that

$$|\chi(k) - 1| < 1 \quad \text{for all } k \in K.$$

For every  $k \in K$ , we then have  $|\chi(k)^n - 1| < 1$  for all  $n \in \mathbf{N}$  and hence  $\chi(k) = 1$ . Therefore  $\chi$  is trivial on  $K$ .

Let  $L$  be the subgroup of  $G$  generated by  $K \cup [H, H]$ . Then  $L$  is a normal and open subgroup of  $H$  and  $\chi$  is trivial on  $L$ . So,  $\chi$  factorizes to a unitary character  $\overline{\chi}$  of the abelian quotient group  $\overline{H} = H/L$ .

Since  $\overline{H}$  is amenable,  $\overline{\chi}$  is weakly contained in the regular representation  $\lambda_{\overline{H}}$  of  $\overline{H}$ , by the Hulanicki-Reiter theorem (see Theorem G.3.2 in [BeHV]). Hence,  $\chi$  is weakly contained in the quasi-regular representation  $\lambda_{H/L}$ , since  $\lambda_{H/L} = \lambda_{\overline{H}} \circ p$ , where  $p : H \rightarrow \overline{H}$  is the quotient homomorphism. By continuity of induction (see Theorem F. 3.5 in [BeHV]), it follows that  $\text{Ind}_H^G \chi$  is weakly contained in

$$\text{Ind}_H^G(\lambda_{H/L}) \cong \lambda_{G/L}. \blacksquare$$

**Proof of Theorem 4**

In view of Theorem 2, it suffices to show that Property  $(T; \mathcal{R}_{\text{quasi-reg}})$  implies  $(T; \mathcal{R}_{\text{mon}})$  for second countable locally compact and totally disconnected groups.

Assume that such a group  $G$  does not have  $(T; \mathcal{R}_{\text{mon}})$ . Then there exists a family  $(H_i, \chi_i)_{i \in I}$  of open subgroups  $H_i$  with unitary characters  $\chi_i$  such that  $1_G$  is weakly contained in the restriction  $\pi'$  of

$$\pi := \bigoplus_{i \in I} \text{Ind}_{H_i}^G \chi_i$$

to the orthogonal complement of the  $\pi(G)$ -invariant vectors. On the other hand, by Lemma 22, there exists a family  $(L_i)_{i \in I}$  of open subgroups  $L_i$  of  $H_i$  such that  $\pi$  is weakly contained in

$$\rho := \bigoplus_{i \in I} \lambda_{G/L_i}.$$

This implies that  $\pi'$  is weakly contained in the restriction of  $\rho$  to the orthogonal complement of the  $\rho(G)$ -invariant vectors. Hence,  $G$  does not have  $(T; \mathcal{R}_{\text{quasi-reg}})$ . ■

**4. PROOF OF THEOREM 8**

(i) The proof is similar to Kazhdan's proof from [Kazh] (see also Lemma 2.14 in [GLMo]): let  $\mathcal{C}$  be the family of open and compactly generated subgroups of  $G$ . Since  $G$  is locally compact,  $1_G$  is weakly contained in the family of quasi-regular representations  $(\lambda_{G/H})_{H \in \mathcal{C}}$ . Hence, by Theorem 2, there exists  $H \in \mathcal{C}$  such that  $G$  has a non-zero invariant vector in  $\ell_2(G/H)$ . This implies that  $H$  has finite index and therefore that  $G$  is compactly generated.

(ii) Assume, by contradiction, that  $G/\overline{[G, G]}$  is not compact. Then there exists a sequence  $(\chi_n)_n$  of unitary characters of  $G$  with  $\chi_n \neq 1_G$  and such that  $\lim_n \chi_n = 1_G$  uniformly on compact subsets of  $G$ . This contradicts Theorem 2.

(iii) • Let  $L$  be a finite index subgroup of  $G$ . We want to show that  $L$  has Property  $(T_{\ell_p})$ .

Let  $\mathcal{L}$  be the set of pairs  $(H, \chi)$  consisting of an open subgroup  $H$  of  $L$  and unitary character  $\chi$  of  $H$ . For  $(H, \chi) \in \mathcal{L}$ , denote by  $\lambda_{(H, \chi)}$  the induced representation  $\text{Ind}_H^L \chi$ . Set

$$\rho = \bigoplus_{(H, \chi) \in \mathcal{L}} \lambda_{(H, \chi)}.$$

Let  $\rho'$  be the restriction of  $\rho$  to the orthogonal complement of the space of  $\rho(L)$ -invariant vectors.

Assume, by contradiction, that  $L$  does not have Property  $(T_{\ell_p})$  for  $p \neq 2$ . Then, by Theorem 2, the trivial representation  $1_L$  of  $L$  is weakly

contained in  $\rho'$ . It follows, by continuity of induction, that  $\lambda_{G/L}$  is weakly contained in  $\text{Ind}_L^G \rho'$ , which is a subrepresentation of

$$\bigoplus_{(H,\chi) \in \mathcal{L}} \text{Ind}_L^G(\lambda_{(H,\chi)}) \cong \bigoplus_{(H,\chi) \in \mathcal{L}} \text{Ind}_H^G \chi.$$

On the other hand,  $1_G$  contained in  $\lambda_{G/L}$ , as  $G/L$  is finite. Therefore,  $1_G$  is weakly contained in  $\text{Ind}_L^G \rho'$ . However,  $\text{Ind}_L^G \rho'$  has no non-zero  $G$ -invariant vector, since  $\rho'$  has no non-zero  $L$ -invariant ones (see [BeHV, Theorem E.3.1]). This is a contradiction to Theorem 2. We conclude that  $L$  has Property  $(T_{\ell_p})$  for  $p \neq 2$ .

• Let  $\tilde{G}$  be a group containing  $G$  as a subgroup of finite index. We want to show that  $\tilde{G}$  has Property  $(T_{\ell_p})$ .

Since  $G$  contains a normal subgroup of  $\tilde{G}$  of finite index and since this subgroup has Property  $(T_{\ell_p})$  for  $p \neq 2$ , by what we have just seen above, we can assume that  $G$  is a normal subgroup of  $\tilde{G}$ .

Assume, by contradiction, that  $\tilde{G}$  does not have Property  $(T_{\ell_p})$  for  $p \neq 2$ . Then there exists an orthogonal representation  $\pi : \tilde{G} \rightarrow \mathbf{O}(\ell_p)$  which has a sequence of almost invariant vectors in the complement  $\ell'_p$  of  $\pi(\tilde{G})$ -invariant vectors in  $\ell_p$ .

Let  $\pi^2$  be the unitary representation of  $\tilde{G}$  on  $\ell_2$  associated to  $\pi$ , as in the beginning of this section. By Lemma 21, there exists a sequence  $(\xi_n)_n$  of almost invariant vectors in the orthogonal complement  $\ell'_2$  of  $\pi^2(\tilde{G})$ -invariant vectors in  $\ell_2$ .

Let  $P : \ell'_2 \rightarrow (\ell'_2)^G$  be the orthogonal projection on the subspace of  $\pi^2(G)$ -invariant vectors in  $\ell'_2$ . Observe that  $(\ell'_2)^G$  is invariant under  $\pi^2(\tilde{G})$ , since  $G$  is normal in  $\tilde{G}$ .

For every  $n \in \mathbf{N}$ , the vector  $\xi_n - P\xi_n$  belongs to the orthogonal complement of  $(\ell'_2)^G$  in  $\ell'_2$ . Hence,  $\xi_n - P\xi_n$  belongs to the orthogonal complement in  $\ell_2$  of the space  $(\ell_2)^G$  of  $G$ -invariant vectors, since  $(\ell_2)^G = (\ell'_2)^G \oplus (\ell_2)^{\tilde{G}}$ . Moreover, we have

$$\lim_n \|\pi^2(g)(\xi_n - P\xi_n) - (\xi_n - P\xi_n)\| = 0, \quad \text{for all } g \in G.$$

It follows that  $\inf_n \|\xi_n - P\xi_n\| = 0$ ; indeed, otherwise,  $\frac{1}{\|\xi_n - P\xi_n\|}(\xi_n - P\xi_n)$  would be a sequence of almost invariant vectors in the orthogonal complement of  $(\ell_2)^G$  in  $\ell_2$  and, using Lemma 21, this would contradict the fact that  $G$  has Property  $(T_{\ell_p})$ . Hence, upon passing to a subsequence, we can assume that  $\lim_n \|\xi_n - P\xi_n\| = 0$ .

Since  $P\xi_n$  is  $G$ -invariant, we have can define the following sequence  $(\eta_n)_n$  of vectors in  $\ell_2$  :

$$\eta_n = \frac{1}{|\tilde{G}/G|} \sum_{t \in \tilde{G}/G} \pi^2(t) P\xi_n.$$

It is clear that  $\eta_n$  is  $\pi^2(\tilde{G})$ -invariant. Moreover, we have

$$\begin{aligned} \|\eta_n - \xi_n\| &\leq \frac{1}{|\tilde{G}/G|} \sum_{t \in \tilde{G}/G} \|\pi^2(t)P\xi_n - \xi_n\| \\ &\leq \frac{1}{|\tilde{G}/G|} \sum_{t \in \tilde{G}/G} \|\pi^2(t)P\xi_n - \pi^2(t)\xi_n\| + \\ &\quad + \frac{1}{|\tilde{G}/G|} \sum_{t \in \tilde{G}/G} \|\pi^2(t)\xi_n - \xi_n\| \\ &= \frac{1}{|\tilde{G}/G|} \sum_{t \in \tilde{G}/G} \|P\xi_n - \xi_n\| + \frac{1}{|\tilde{G}/G|} \sum_{t \in \tilde{G}/G} \|\pi^2(t)\xi_n - \xi_n\|. \end{aligned}$$

It follows that  $\lim_n \|\eta_n - \xi_n\| = 0$ . Hence,  $\eta_n \neq 0$  for sufficiently large  $n$ , since  $\|\xi_n\| = 1$ .

For every  $t \in \tilde{G}$ , the vector  $\pi^2(t)(P\xi_n)$  belongs to  $(\ell'_2)^G$ , since  $(\ell'_2)^G$  is invariant under  $\pi^2(\tilde{G})$ . It follows that  $\eta_n \in (\ell'_2)^G$  and, in particular,  $\eta_n \in \ell'_2$ . This is a contradiction, as there are no non-zero  $\pi^2(\tilde{G})$ -invariant vector in  $\ell'_2$ .

(iv) Since  $G$  is totally disconnected, we can find a compact open subgroup  $K$  of  $G$ , by van Dantzig's theorem. The amenability of  $G$  implies the amenability of its action on  $G/K$ :  $1_G$  is weakly contained in  $\lambda_{G/K}$  (see Théorème on p. 28 in [Eyma]). As  $G$  has Property  $(T_{\ell_p})$ , it follows from Theorem 4 that  $G$  has a non-zero invariant vector in  $\ell_2(G/K)$ . Hence,  $K$  has finite index in  $G$  and  $G$  is compact. ■

### Proof of Theorem 8

Since  $G$  has the Howe-Moore property, every proper open subgroup of  $G$  is compact (see [CCLTV, Proposition 3.2]). It follows that, for every proper open subgroup  $H$ , the space  $\ell_2(G/H)$  can be identified with the  $G$ -invariant subspace of  $L_2(G)$  of functions on  $G$  which are right  $H$ -invariant. As a consequence, we see that  $\lambda_{G/H}$  is a subrepresentation of the regular representation  $\lambda_G$ . Denoting by  $\mathcal{L}$  be the set of proper open subgroups of  $G$ , this implies that  $\bigoplus_{H \in \mathcal{L}} \lambda_{G/H}$  is weakly contained in the regular representation  $\lambda_G$ .

On the other hand, since  $G$  is not amenable,  $1_G$  is not weakly contained in  $\lambda_G$ , by the Reiter-Hulanicki theorem. It follows that  $1_G$  is not weakly contained in  $\bigoplus_{H \in \mathcal{L}} \lambda_{G/H}$  and Theorem 4 shows that  $G$  has Property  $(T_{\ell_p})$  for  $p \neq 2$ . ■

## 5. DISCRETE GROUPS WITH PROPERTY $(T_{\ell_p})$

The main tool for the proof of Theorem 12 is the following rigidity result concerning the unitary representations of the groups appearing in the statement.

Given locally compact second countable groups  $G$  and  $Q$  and a continuous homomorphism  $f : G \rightarrow Q$  with dense image, let us say as in [Corn1, Definition 4.2.2] that  $f$  is a *resolution* if, whenever a unitary representation  $\pi$  of  $G$  has a sequence of almost invariant vectors,  $\pi$  has a non-zero subrepresentation  $\rho$  which factors through a unitary representation  $\tilde{\rho}$  of  $Q$ , that is,  $\rho = \tilde{\rho} \circ f$ .

A weaker form (which is however often sufficient for the applications) of the following theorem was established in [LuZi, Theorem 2.2]; for more general versions of it, see [Marg, Chap. III, Theorem 6.3], [BeLo, Theorem A], and [Corn1, Theorem 4.3.1].

**Theorem 23.** *Let  $G_1$  and  $G_2$  be second countable locally compact groups and  $\Gamma$  an irreducible lattice in  $G = G_1 \times G_2$ . Assume that  $G_1$  has Property (T). Then the canonical projection  $p_2 : \Gamma \rightarrow G_2$  is a resolution. ■*

The second ingredient for the proof of Theorem 12 is the following elementary lemma.

**Lemma 24.** *Let  $G$  be a connected topological group and  $\Gamma$  a dense subgroup of  $G$ . Let  $H_1, \dots, H_n$  be subgroups of  $\Gamma$  such that no  $H_i$  is dense in  $G$ . Let  $X = \bigcup_{i=1}^n a_i H_i$  be a union of cosets of the  $H_i$ 's. Then  $\Gamma \setminus X$  is dense in  $G$ .*

**Proof** Assume, by contradiction, that  $\Gamma \setminus X$  is not dense in  $G$ . Then there exists a non-empty open subset  $U$  of  $G$  such that  $U \cap (\Gamma \setminus X) = \emptyset$  and hence

$$U \cap \overline{(\Gamma \setminus X)} = \emptyset.$$

Since  $\Gamma$  is dense in  $G$ , it follows that  $U$  is contained in

$$\overline{X} = \bigcup_{i=1}^n a_i \overline{H_i}.$$

This implies that some coset  $a_i \overline{H_i}$  has non-empty interior in  $G$ . Thus,  $\overline{H_i}$  is open in  $G$  and hence  $\overline{H_i} = G$ , since  $G$  is connected. This is a contradiction. ■

### Proof of Theorem 12

Assume, by contradiction, that  $\Gamma$  does not have Property  $(T_{\ell_p})$ . By Theorem 4, there exists a family  $\mathcal{L}$  of subgroups of  $\Gamma$  such that  $1_\Gamma$  is weakly contained in the restriction  $\pi'$  of

$$\pi = \bigoplus_{H \in \mathcal{L}} \lambda_{\Gamma/H}$$

to the orthogonal complement the space of  $\pi(\Gamma)$ -invariant vectors.

Now,  $\Gamma$  has Property  $(\tau)$ ; see Remark 13. So, we can assume that every  $H \in \mathcal{L}$  has infinite index in  $\Gamma$ . In particular,  $\pi'$  coincides with  $\pi$  and acts on  $\mathcal{H} = \bigoplus_{H \in \mathcal{L}} \ell_2(\Gamma/H)$ .

It follows from Theorem 23 that there exists a non-zero  $\Gamma$ -invariant subspace  $\mathcal{K}$  of  $\mathcal{H}$  such that the corresponding  $\Gamma$ -representation factors through a (continuous) unitary representation of  $G_2$ .

Let  $H \in \mathcal{L}$  be such that the orthogonal projection  $P(\mathcal{K})$  of  $\mathcal{K}$  on  $\ell_2(\Gamma/H)$  is non-zero. Then the restriction of  $\pi$  to  $P(\mathcal{K})$  also factors through a unitary representation  $\tilde{\pi}$  of  $G_2$  (see [Corn1, Lemma 4.1.5]).

We claim that  $p_2(H)$  is not dense in  $G_2$ . Indeed, assume that this is not the case. The representation of  $G$ , obtained by inducing the restriction of  $\pi$  to  $P(\mathcal{K})$ , is a subrepresentation of

$$\text{Ind}_{\Gamma}^G(\lambda_{\Gamma/H}) \cong \lambda_{G/H}.$$

Since  $\Gamma$  is a lattice, it follows that there exists a non-zero  $G_1$ -invariant vector  $f \in L_2(G/H)$ . Lifting  $f$  to  $G$ , we obtain a measurable function  $f : G \rightarrow \mathbb{C}$  such that, for every  $g \in G_1$ ,  $f(gxh) = f(x)$  for almost every  $x \in G$  and every  $h \in H$ . Upon changing  $f$  on a null set, we can assume that the previous equality holds for all  $g \in G_1$ ,  $x \in G$  and all  $h \in H$  (see [Zimm, Lemma 2.2.16]). Thus,  $f$  is invariant under right translation by elements from  $G_1H$ . Since, by assumption,  $G_1H$  is dense in  $G$ , it follows that  $f$  is constant, up to a null set (see [Zimm, Lemma 2.2.13]). As  $f \in L_2(G/H)$  is non zero, this implies that  $G/H$  has finite volume. This is impossible, since  $H$  has infinite index in  $\Gamma$ . So,  $p_2(H)$  is not dense in  $G_2$ .

Let  $f$  be a unit-vector in  $P(\mathcal{K})$ . We claim that we can find a sequence  $\gamma_n \in \Gamma$  with

$$\lim_n p_2(\gamma_n) = e \quad \text{and} \quad \lim_n \langle \lambda_{\Gamma/H}(\gamma_n)f, f \rangle = 0. \quad (**)$$

This will yield the desired contradiction, since

$$\lim_n \langle \lambda_{\Gamma/H}(\gamma_n)f, f \rangle = \lim_n \langle \tilde{\pi}(p_2(\gamma_n))f, f \rangle = 1,$$

by continuity of  $\tilde{\pi}$ .

Let  $\varepsilon > 0$ . There exists a finite subset  $F$  of  $\Gamma/H$  such that

$$\sum_{x \notin F} |f(x)|^2 \leq \varepsilon.$$

Choose a set  $A \in \Gamma$  of representatives for the cosets  $x \in F$ . Set

$$X = \bigcup_{a,b \in A} p_2(b)p_2(H)p_2(a^{-1}) = \bigcup_{a,b \in A} p_2(ba^{-1})p_2(H^a),$$

which is a finite union of cosets of subgroups conjugate to  $p_2(H)$ .

As we have shown above,  $p_2(H)$  is not dense in  $G_2$ ; the same is true for its conjugate subgroups. Since  $G_2$  is connected, it follows from Lemma 24 that  $p_2(\Gamma) \setminus X$  is dense in  $G_2$ . We can therefore find a sequence  $(\gamma_n)_n$  in  $\Gamma$  with  $\lim_n p_2(\gamma_n) = e$  such that, for all  $n$ ,

$$p_2(\gamma_n)p_2(a)p_2(H) \cap p_2(b)p_2(H) = \emptyset, \quad \text{for all } a, b \in A,$$

and hence

$$\gamma_n aH \cap bH = \emptyset, \quad \text{for all } a, b \in A,$$

for all  $n$ . We then have

$$\begin{aligned} |\langle \lambda_{\Gamma/H}(\gamma_n)f, f \rangle| &\leq \sum_{x \in F} |f(\gamma_n^{-1}x)f(x)| + \sum_{x \notin F} |f(\gamma_n^{-1}x)f(x)| \\ &\leq 2\|f\| \left( \sum_{x \notin F} |f(x)|^2 \right)^{1/2} \\ &\leq 2\sqrt{\varepsilon} \end{aligned}$$

It follows that the claim  $(**)$  is satisfied by a subsequence of  $(\gamma_n)_n$ . ■

### Proof of Proposition 15

Assume that  $\Gamma$  does not have Property  $(T_{\ell_p})$ . Let  $\mathcal{L}$  be the family of proper subgroups of  $\Gamma$  and  $X = \coprod_{H \in \mathcal{L}} \Gamma/H$ . Theorem 4 shows that the action of  $\Gamma$  on  $X$  is amenable. However,  $\Gamma$  has no fixed point in  $X$ . Hence,  $\Gamma$  does not have Property  $(F)$ . ■

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